

Let $ABCD$ be a square. Construct equilateral triangle PBA inside it. I is midpoint of AB . M, N lie on PA, PB such that triangle MIN is right isosceles at I . K, L are nine point center of triangles PBC, PAD . Prove that $KLMN$ is a rectangle.

Solution by Takis Chronopoulos

(10 January 2017, fixed mistypes at 14 January)

Lemma : $\angle DCP = 15^\circ$

Proof of the Lemma

(proven as problem 010 ,

<https://web.facebook.com/groups/parmenides52/permalink/1026833390763700/>)

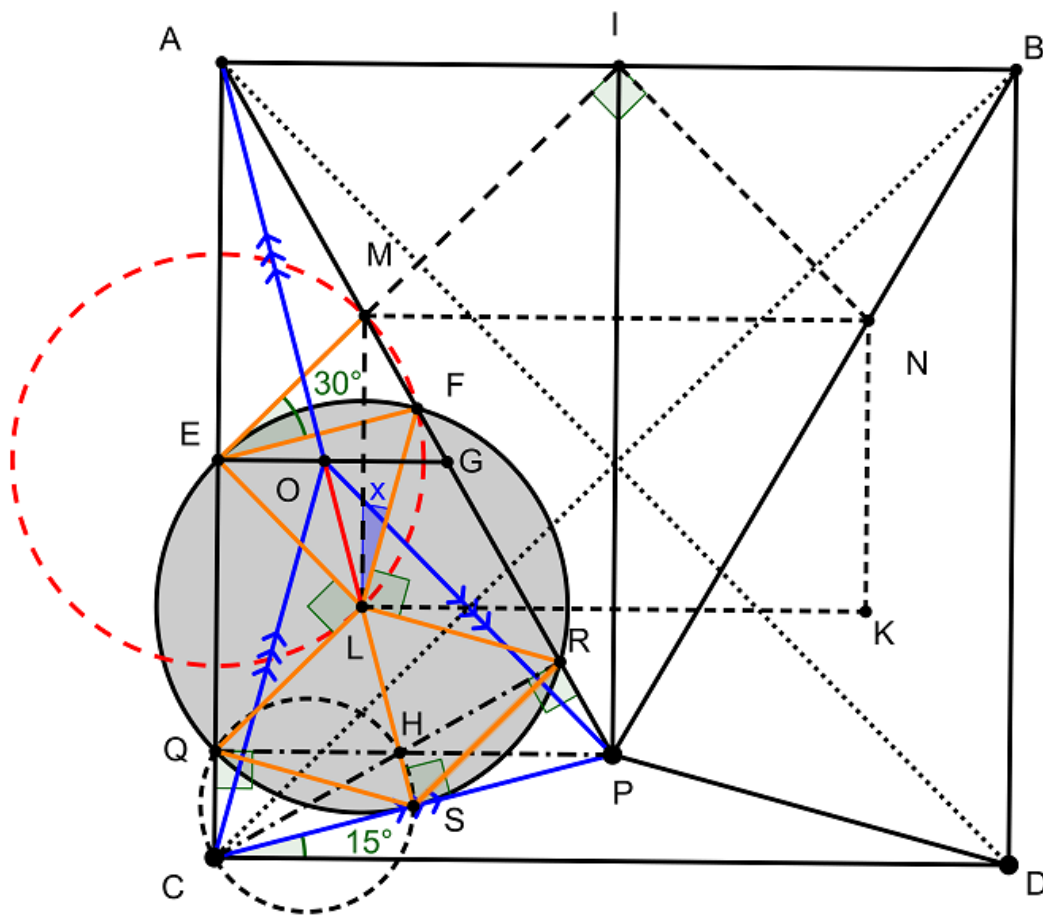
Because IP is line of symmetry of the figure $MN \parallel AB$, $LK \parallel AB$
 $\Rightarrow MN \parallel LK \Rightarrow MNKL$ is a trapezoid or a parallelogram

In order $MNKL$ to be rectangle it is sufficient to prove that $ML \parallel AC$
 (then similarly $KN \parallel BD$ and since $AC \parallel BD$, it would follow that $ML \parallel KN$ and since $AC \perp AB$, it would follow that $ML \perp MN$)

Let E, F be the midpoints of AC, AP respectively,
 Let PQ, CR, AS be the altitudes of the $\triangle APC$

(Lemma) $\angle PCD = 15^\circ \Rightarrow \angle ACP = \angle APC = 75^\circ$ (1)
 $\Rightarrow \triangle APC$ isosceles \Rightarrow median $AS \equiv$ Euler line of $\triangle APC$

Let O, H be the circumcenter, orthocenter of $\triangle APC$ respectively \Rightarrow Nine Point Circle Center L of $\triangle APC$ is the midpoint of OH
 $\Rightarrow LE = LF = LQ = LR = LS = R_9$ (2)



O is the circumcenter of $\triangle APC \Rightarrow \angle CAO = \angle OCA = \angle OPC = 15^\circ$ (3) $\Rightarrow \angle OCP = 75^\circ - 15^\circ = 60^\circ$ and because of (2) $\Rightarrow \triangle COP$ is equilateral

$\triangle CRP$: $\angle P = 90^\circ$, $\angle RPC = 75^\circ$ and because of (1) $\Rightarrow \angle HCS = 15^\circ \Rightarrow \angle QCH = 75^\circ - 15^\circ = 60^\circ$ (4)

$\angle CQP + \angle HSC = 90^\circ + 90^\circ = 180^\circ \Rightarrow CQHS$ cyclic, (4) $\Rightarrow \angle HSQ = \angle QCH = 60^\circ$ and because of (2) $\Rightarrow \triangle QLS$ is equilateral

$\Rightarrow CQHS$ cyclic $\Rightarrow \angle CQS = \angle CHS = 90^\circ - 15^\circ = 75^\circ$

$\Rightarrow \angle EQL = 180^\circ - 60^\circ - 75^\circ = 45^\circ$ and because of (2)

$\Rightarrow \triangle ELQ$ right isosceles

$\Rightarrow \angle ELQ = 90^\circ$, $\angle QEL = 45^\circ$ (5)

Similarly because AS is the line of symmetry of $\triangle ACP$

$\Rightarrow \angle LFR = 45^\circ$, $\angle FLR = 90^\circ$ and $\triangle RLS$ is equilateral

So $\angle ELF = 360^\circ - 90^\circ - 90^\circ - 60^\circ - 60^\circ = 60^\circ$

and because of (2) $\Rightarrow \triangle EFL$ is equilateral

$\Rightarrow \angle ELF = \angle FEL = \angle EFL = 60^\circ$, $EF = EL = FL$ (6)

Obviously $\angle AIE = \angle AEI = 45^\circ$, (5), (6) \Rightarrow

$\Rightarrow \angle MEF = 180^\circ - 45^\circ - 60^\circ - 45^\circ = 30^\circ$ (7)

E, F midpoints of AC, AP respectively $\Rightarrow EF \parallel CP$

$\Rightarrow \angle MFE = \angle APC = 75^\circ$ (8)

$\triangle MFE$: (7), (8) $\Rightarrow \angle EMF = 180^\circ - 30^\circ - 75^\circ = 75^\circ = \angle MFE$

$\Rightarrow \triangle MFE$ isosceles $\Rightarrow ME = MF$ and because of (6)

$\Rightarrow ME = EF = EL \Rightarrow E$ is the circumcenter of $\triangle MFL$

$\Rightarrow \angle MLF = \angle MEF / 2 = 30^\circ / 2 = 15^\circ$

$\Rightarrow \angle MLO = \angle OLF - \angle MLF = 30^\circ - 15^\circ = 15^\circ$ and because of (3) $\Rightarrow ML \parallel AC$, qed

